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1995 J. Phys. A: Math. Gen. 28 3533

(http://iopscience.iop.org/0305-4470/28/12/024)

J. Phys. A: Math. Gen. 28 (1995) 3533–3540. Printed in the UK

Knizhnik–Zamolodchikov equations and the Calogero–Sutherland–Moser integrable models with exchange terms

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Received 9 March 1995

Abstract. It is shown that from some solutions of generalized Knizhnik–Zamolodchikov equations one can construct eigenfunctions of the Calogero–Sutherland–Moser Hamiltonians with exchange terms, which are characterized by any given permutational symmetry under particle exchange. This generalizes some results previously derived by Matsuo and Cherednik for the ordinary Calogero–Sutherland–Moser Hamiltonians.

1. *Introduction*

Recently, much attention has been paid to the Calogero–Sutherland–Moser (CSM) integrable systems [1–3] both in field-theoretical and in condensed-matter contexts. They are indeed relevant to several apparently disparate physical problems, such as fractional statistics and anyons [4], spin chain models [5], soliton wave propagation [6], two-dimensional non-perturbative quantum gravity and string theory [7], and two-dimensional QCD [8].

Such one-dimensional integrable systems consist of N non-relativistic particles interacting through two-body potentials of the inverse square type and its generalizations, and are related to root systems of A_{N-1} algebras [9]. Their spectra and wavefunctions can be obtained by simultaneously diagonalizing a set of N commuting first-order differential operators, first considered by Dunkl in the mathematical literature [10], and later rediscovered by Polychronakos [11] and Brink *et al* [12]. The use of Dunkl operators leads to Hamiltonians with exchange terms, related to the spin generalizations of the CSM models [13].

Dunkl operators are rather similar [14] to the differential operators of the Knizhnik–Zamolodchikov (KZ) equations, which first appeared in conformal field theory [15]. Matsuo [16] and Cherednik [17] proved that from some solutions of the KZ equations, one can construct wavefunctions for the (ordinary) CSM systems. Such relations between KZ equations and CSM systems were then exploited by Felder and Veselov [18] to provide a natural interpretation for the shift operators of the latter.

The purpose of the present paper is to extend the results of Matsuo and Cherednik to some CSM models with exchange terms. In the following section, we review generalized KZ equations. Then, in section 3, we establish new links between some of their solutions

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0305-4470/95/123533+08$19.50

and wavefunctions of corresponding CSM models with exchange terms. Finally, section 4 contains the conclusion.

2. Generalized Knizhnik–Zamolodchikov equations

Let us consider a system of N first-order partial differential equations of the type

$$\partial_i \Phi = \left(\sum_{j \neq i} \left(f_{ij}(x_i - x_j) P^{(ij)} + c\, T^{(ij)} \right) + \lambda^{(i)} \right) \Phi \qquad i = 1, 2, \ldots, N \tag{1}$$

where $\Phi = \Phi(x_1, x_2, \ldots, x_N)$ takes values in the tensor product $V \otimes V \otimes \cdots \otimes V = V^{\otimes N}$ of some N-dimensional vector space V, $f_{ij}(x_i - x_j)$ is a function of $x_i - x_j$ and c is a complex parameter. Equation (1) also contains three operators $P^{(ij)}$, $T^{(ij)}$ and $\lambda^{(i)}$, defined as in [18], i.e. $\lambda^{(i)}$ is the operator in $V^{\otimes N}$ acting on the ith factor as the diagonal matrix $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and identically on all other factors, P is the permutation: $P(a \otimes b) = b \otimes a$, T is the following operator on $V \otimes V$:

$$T = \sum_{k>l} (E_{kl} \otimes E_{lk} - E_{lk} \otimes E_{kl}) \tag{2}$$

where E_{kl} denotes the $N \times N$ matrix with entry 1 in row k and column l and zeros everywhere else and $P^{(ij)}$ and $T^{(ij)}$ are the corresponding operators in $V^{\otimes N}$ acting only on the ith and jth factors. When $\lambda_l = c = 0$ and $f_{ij}(x_i - x_j) = k(x_i - x_j)^{-1}$, the set of equations (1) coincides with that derived by Knizhnik and Zamolodchikov in conformal field theory [15]. We shall therefore refer to (1) as generalized KZ equations.

Let us consider the case where Φ has the form

$$\Phi = \sum_{\sigma \in S_N} \Phi_\sigma e_\sigma \qquad e_\sigma = e_{\sigma(1)} \otimes e_{\sigma(2)} \otimes \cdots \otimes e_{\sigma(N)} \tag{3}$$

where S_N is the symmetric group and e_k denotes a column vector with entry 1 in row k and zeros everywhere else. The operators $P^{(ij)}$, $T^{(ij)}$ and $\lambda^{(i)}$ transform the components Φ_σ into $\Phi_{\sigma \circ p_{ij}}$, $\tau_\sigma^{(ij)} \Phi_{\sigma \circ p_{ij}}$ and $\lambda_{\sigma(i)} \Phi_\sigma$, respectively, where $p_{ij} \in S_N$ is the transposition of i and j and $\tau_\sigma^{(ij)} \equiv \mathrm{sgn}\big(\sigma(i) - \sigma(j)\big)$ satisfies the relations

$$\begin{gathered} \tau_\sigma^{(ij)} = -\tau_{\sigma \circ p_{ij}}^{(ij)} = -\tau_\sigma^{(ji)} \qquad \tau_{\sigma \circ p_{ij}}^{(ik)} = \tau_\sigma^{(jk)} \qquad \tau_{\sigma \circ p_{ij}}^{(kl)} = \tau_\sigma^{(kl)} \\ \tau_\sigma^{(ij)} \tau_\sigma^{(ik)} + \tau_\sigma^{(jk)} \tau_\sigma^{(ji)} + \tau_\sigma^{(kl)} \tau_\sigma^{(kj)} = 1 \end{gathered} \tag{4}$$

for any $i \neq j \neq k \neq l$. Hence, for such functions Φ, equation (1) is equivalent to the set of equations

$$\partial_i \Phi_\sigma = \sum_{j \neq i} \left(f_{ij}(x_i - x_j) + c\, \tau_\sigma^{(ij)} \right) \Phi_{\sigma \circ p_{ij}} + \lambda_{\sigma(i)} \Phi_\sigma \qquad i = 1, 2, \ldots, N \tag{5}$$

where σ is an arbitrary permutation of S_N.

The integrability conditions of (5), i.e. $\partial_j \partial_i \Phi_\sigma = \partial_i \partial_j \Phi_\sigma$ for any i, $j = 1, 2, \ldots, N$, and any $\sigma \in S_N$, are satisfied if and only if

$$f_{ij}(x_i - x_j) = -f_{ji}(x_j - x_i) \tag{6}$$

$$f_{ij}(x_i - x_j) f_{jk}(x_j - x_k) + f_{jk}(x_j - x_k) f_{ki}(x_k - x_i) + f_{ki}(x_k - x_i) f_{ij}(x_i - x_j) = -c^2 \tag{7}$$

for any i, j, $k = 1, 2, \ldots, N$, such that $i \neq j \neq k$. By taking (6) into account, equation (7) can be rewritten as

$$f_{ij}(u) f_{jk}(v) - f_{ik}(u + v) \left[f_{ij}(u) + f_{jk}(v) \right] = -c^2. \tag{8}$$

It is sufficient to consider the latter for $1 \leqslant i < j < k \leqslant N$, since the relations corresponding to different orderings of i, j, k directly follow from them.

Equation (8) looks like a functional equation first considered by Sutherland [2] and solved by Calogero [19] through a small-x expansion. By using a similar procedure, all the solutions of (8) that are odd and meromorphic in a neighbourhood of the origin can easily be derived. Denote by $F(u)$ and $G(u)$ the functions

$$F(u) = \begin{cases} k\omega \coth \omega u & \text{if } c^2 = k^2\omega^2 > 0 \\ k/u & \text{if } c^2 = 0 \\ k\omega \cot \omega u & \text{if } c^2 = -k^2\omega^2 < 0 \end{cases} \tag{9}$$

and

$$G(u) = \begin{cases} k\omega \tanh \omega u & \text{if } c^2 = k^2\omega^2 > 0 \\ -k\omega \tan \omega u & \text{if } c^2 = -k^2\omega^2 < 0 \end{cases} \tag{10}$$

where $\omega \in \mathbb{R}^+$. For any $N \geqslant 3$ and $c^2 \neq 0$, one finds that equation (8) has two and only two types of odd, meromorphic solutions, namely

$$f_{ij}(u) = f_{ji}(u) = F(u) \qquad 1 \leqslant i < j \leqslant N \tag{11}$$

and

$$f_{ij}(u) = f_{ji}(u) = \begin{cases} F(u) & \text{if } 1 \leqslant i < j \leqslant N_1 \text{ or } N_1 + 1 \leqslant i < j \leqslant N \\ G(u) & \text{if } 1 \leqslant i \leqslant N_1 \text{ and } N_1 + 1 \leqslant j \leqslant N \end{cases} \tag{12}$$

where in (12), N_1 may take any value in the set $\{1, 2, \ldots, N-1\}$. Moreover, for any $N \geqslant 3$ and $c^2 = 0$, equation (8) has one and only one odd, meromorphic solution, given by (11). Both solutions (11) and (12) are well known and describe either particles of the same type or of two different types [19].

It should be noted that equation (8) also has some singular solutions, such as

$$f_{ij}(u) = f_{ji}(u) = c\,\mathrm{sgn}(u) = c\,[\theta(u) - \theta(-u)] \qquad 1 \leqslant i < j \leqslant N \tag{13}$$

where $\theta(u)$ denotes the Heaviside function.

3. Solutions of Calogero–Sutherland–Moser models with exchange terms

From a set of $N!$ functions $\Phi_\sigma(x_1, \ldots, x_N)$, $\sigma \in S_N$, satisfying equation (5), one can in general construct $N!$ functions $\varphi_{rs}^{[f]}(x_1, \ldots, x_N)$, defined by

$$\varphi_{rs}^{[f]} = \sum_{\sigma \in S_N} V_{rs}^{[f]}(\sigma) \Phi_\sigma \tag{14}$$

where $[f] \equiv \left[f_1 f_2 \ldots f_N\right]$ runs over all N-box Young diagrams, r and s label the standard tableaux associated with $[f]$, arranged in lexicographical order, and $V_{rs}^{[f]}(\sigma)$ denotes Young's orthogonal matrix representation of S_N [20]. Such functions $\varphi_{rs}^{[f]}$ satisfy the system of equations

$$\begin{aligned} \partial_i \varphi_{rs}^{[f]} = {} & \sum_{j \neq i} f_{ij} \sum_t \varphi_{rt}^{[f]} V_{ts}^{[f]}(p_{ij}) - c \sum_{j \neq i} \sum_t \left(\sum_\sigma \tau_\sigma^{(ij)} V_{rt}^{[f]}(\sigma) \Phi_\sigma \right) V_{ts}^{[f]}(p_{ij}) \\ & + \sum_\sigma \lambda_{\sigma(i)} V_{rs}^{[f]}(\sigma) \Phi_\sigma \qquad i = 1, 2, \ldots, N. \end{aligned} \tag{15}$$

In deriving (15), use has been made of the first of the following representation properties of $V^{[f]}_{rs}(\sigma)$:

$$V^{[f]}_{rs}(\sigma \circ \sigma') = \sum_t V^{[f]}_{rt}(\sigma) V^{[f]}_{ts}(\sigma') \qquad V^{[f]}_{rs}(1) = \delta_{r,s} \tag{16}$$

and of the first equality in (4).

From (15), one has that

$$\begin{aligned}
\partial^2_{ii}\varphi^{[f]}_{rs} = {} & \sum_{j\neq i}(\partial_i f_{ij}) \sum_t \varphi^{[f]}_{rt} V^{[f]}_{ts}(p_{ij}) + \sum_{j\neq i} f_{ij} \sum_t \left(\partial_i \varphi^{[f]}_{rt}\right) V^{[f]}_{ts}(p_{ij}) \\
& - c \sum_{j\neq i} \sum_t \left(\sum_\sigma \tau^{(ij)}_\sigma V^{[f]}_{rt}(\sigma) \partial_i \Phi_\sigma \right) V^{[f]}_{ts}(p_{ij}) \\
& + \sum_\sigma \lambda_{\sigma(i)} V^{[f]}_{rs}(\sigma) \partial_i \Phi_\sigma .
\end{aligned} \tag{17}$$

By using (4), (5), (15) and (16) again, and by summing over i, we obtain the following result for the Laplacian of $\varphi^{[f]}_{rs}$:

$$\begin{aligned}
\Delta\varphi^{[f]}_{rs} = {} & \varphi^{[f]}_{rs} \left(\sum_{\substack{i,j\\ i\neq j}} (f^2_{ij} - c^2) + \sum_i \lambda^2_i \right) \\
& + \sum_t \varphi^{[f]}_{rt} \left(\sum_{\substack{i,j\\ i\neq j}} (\partial_i f_{ij}) V^{[f]}_{ts}(p_{ij}) + \sum_{\substack{i,j,k\\ i\neq j\neq k}} f_{ij} f_{ik} V^{[f]}_{ts}(p_{ik} \circ p_{ij}) \right) \\
& - c \sum_\sigma \sum_t V^{[f]}_{rt}(\sigma) \left(\sum_{\substack{i,j,k\\ i\neq j\neq k}} \left(f_{ij} \tau^{(ik)}_\sigma + f_{ik} \tau^{(kj)}_\sigma \right) V^{[f]}_{ts}(p_{ik} \circ p_{ij}) \right) \Phi_\sigma \\
& + \sum_\sigma \sum_t V^{[f]}_{rt}(\sigma) \left(\sum_{\substack{i,j\\ i\neq j}} (\lambda_{\sigma(i)} + \lambda_{\sigma(j)}) f_{ij} V^{[f]}_{ts}(p_{ij}) \right) \Phi_\sigma \\
& + c^2 \sum_\sigma \sum_t V^{[f]}_{rt}(\sigma) \left(\sum_{\substack{i,j,k\\ i\neq j\neq k}} \tau^{(ik)}_\sigma \tau^{(kj)}_\sigma V^{[f]}_{ts}(p_{ik} \circ p_{ij}) \right) \Phi_\sigma \\
& - c \sum_\sigma \sum_t V^{[f]}_{rt}(\sigma) \left(\sum_{\substack{i,j\\ i\neq j}} (\lambda_{\sigma(i)} + \lambda_{\sigma(j)}) \tau^{(ij)}_\sigma V^{[f]}_{ts}(p_{ij}) \right) \Phi_\sigma .
\end{aligned} \tag{18}$$

We shall now proceed to evaluate the various terms on the right-hand side of (18).

As

$$p_{ik} \circ p_{ij} = p_{ij} \circ p_{jk} = p_{jk} \circ p_{ik} \tag{19}$$

the last part of the second term becomes

$$\sum_{\substack{i,j,k\\ i\neq j\neq k}} f_{ij} f_{ik} V_{ts}^{[f]}(p_{ik}\circ p_{ij})$$

$$= \sum_{\substack{i,j,k\\ i<j<k}} (f_{ij}f_{ik} + f_{jk}f_{ji} + f_{ki}f_{kj})\Big(V_{ts}^{[f]}(p_{ik}\circ p_{ij}) + V_{ts}^{[f]}(p_{ij}\circ p_{ik})\Big)$$

$$= c^2 \sum_{\substack{i,j,k\\ i<j<k}} \Big(V_{ts}^{[f]}(p_{ik}\circ p_{ij}) + V_{ts}^{[f]}(p_{ij}\circ p_{ik})\Big) \tag{20}$$

where in the last step we used the integrability conditions (6) and (7) of (5). By applying (19) again, the summation over i, j, k in the third term on the right-hand side of (18) can be rewritten as

$$\sum_{\substack{i,j,k\\ i\neq j\neq k}} \Big(f_{ij}\tau_\sigma^{(ik)} + f_{ik}\tau_\sigma^{(kj)}\Big) V_{ts}^{[f]}(p_{ik}\circ p_{ij})$$

$$= \sum_{\substack{i,j,k\\ i<j<k}} \Big(\Big((f_{ij}+f_{ji})\tau_\sigma^{(ik)} + (f_{jk}+f_{kj})\tau_\sigma^{(ji)} + (f_{ki}+f_{ik})\tau_\sigma^{(kj)}\Big) V_{ts}^{[f]}(p_{ik}\circ p_{ij})$$

$$+\Big((f_{ij}+f_{ji})\tau_\sigma^{(jk)} + (f_{jk}+f_{kj})\tau_\sigma^{(ki)} + (f_{ki}+f_{ik})\tau_\sigma^{(ij)}\Big) V_{ts}^{[f]}(p_{ij}\circ p_{ik})\Big) \tag{21}$$

and therefore vanishes owing to the antisymmetry of f_{ij} in i, j, as shown in (6). The same is true for the summations over i, j in the fourth and sixth terms as a consequence of the antisymmetry of f_{ij} and $\tau_\sigma^{(ij)}$, respectively. Finally, by successively using (19) and (4), the summation over i, j, k in the fifth term becomes

$$\sum_{\substack{i,j,k\\ i\neq j\neq k}} \tau_\sigma^{(ik)}\tau_\sigma^{(kj)} V_{ts}^{[f]}(p_{ik}\circ p_{ij})$$

$$= \sum_{\substack{i,j,k\\ i<j<k}} \Big(\Big(\tau_\sigma^{(ji)}\tau_\sigma^{(ik)} + \tau_\sigma^{(kj)}\tau_\sigma^{(ji)} + \tau_\sigma^{(ik)}\tau_\sigma^{(kj)}\Big) V_{ts}^{[f]}(p_{ik}\circ p_{ij})$$

$$+\Big(\tau_\sigma^{(ki)}\tau_\sigma^{(ij)} + \tau_\sigma^{(ij)}\tau_\sigma^{(jk)} + \tau_\sigma^{(jk)}\tau_\sigma^{(ki)}\Big) V_{ts}^{[f]}(p_{ij}\circ p_{ik})\Big)$$

$$= -\sum_{\substack{i,j,k\\ i<j<k}} \Big(V_{ts}^{[f]}(p_{ik}\circ p_{ij}) + V_{ts}^{[f]}(p_{ij}\circ p_{ik})\Big). \tag{22}$$

By putting all results together, the Laplacian of $\varphi_{rs}^{[f]}$ takes the simple form

$$\Delta\varphi_{rs}^{[f]} = \left(\sum_{\substack{i,j\\ i\neq j}} \big(f_{ij}^2(x_i-x_j) + \big(\partial_i f_{ij}(x_i-x_j)\big) K_{ij} - c^2\big) + \sum_i \lambda_i^2\right) \varphi_{rs}^{[f]} \tag{23}$$

where $K_{ij} = K_{ji}$, $1 \leqslant i < j \leqslant N$, are some operators, whose action on $\varphi_{rs}^{[f]}$ is defined by

$$K_{ij}\varphi_{rs}^{[f]} = \sum_t \varphi_{rt}^{[f]} V_{ts}^{[f]}(p_{ij}). \tag{24}$$

Let us emphasize that equation (23) is valid for any function $\varphi_{rs}^{[f]}$ constructed from any solution of (5) via transformation (14).

In the special cases where $[f] = [N]$ or $[1^N]$, since $V^{[N]}(p_{ij}) = -V^{[1^N]}(p_{ij}) = 1$, the operators K_{ij} behave as I or $-I$, respectively. Hence $\varphi^{[N]} = \sum_\sigma \Phi_\sigma$ and $\varphi^{[1^N]} = \sum_\sigma (-1)^\sigma \Phi_\sigma$, where $(-1)^\sigma$ is the parity of permutation σ, are eigenfunctions of the operators $-\Delta + \sum_{i \neq j}(f_{ij}^2 \pm \partial_i f_{ij} - c^2)$, where the upper (lower) sign corresponds to the former (latter). For f_{ij} given in (11), these essentially fit the results of Matsuo [16] and Cherednik [17].

In the mixed symmetry cases where $[f] \neq [N]$, $[1^N]$, the operators K_{ij} have a non-trivial effect on the functions $\varphi_{rs}^{[f]}$. Provided the latter satisfy the conditions

$$\begin{aligned}\varphi_{rs}^{[f]}&(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) \\ &= \sum_t \varphi_{rt}^{[f]}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) V_{ts}^{[f]}(p_{ij}) \qquad 1 \leqslant i < j \leqslant N \end{aligned}\tag{25}$$

which amount to

$$\begin{aligned}\Phi_\sigma&(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N) \\ &= \Phi_{\sigma \circ p_{ij}}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) \qquad 1 \leqslant i < j \leqslant N\end{aligned}\tag{26}$$

for any $\sigma \in S_N$, the operators K_{ij} may be interpreted as permutation operators acting on the variables x_i and x_j,

$$K_{ij} x_j = x_i K_{ij} \qquad K_{ij} x_k = x_k K_{ij} \qquad k \neq i, j. \tag{27}$$

It remains for us to examine under which conditions equation (5) admits solutions satisfying (26). This is readily done by differentiating both sides of (26) with respect to x_k and using (5) to calculate the derivatives. Equations (5) and (26) are found to be compatible if and only if all functions $f_{ij}(u)$, $i \neq j$, coincide, hence in cases such as (11) and (13). For the former choice, equation (23) becomes

$$\left(-\Delta + \omega^2 \sum_{\substack{i,j \\ i \neq j}} \left(\operatorname{csch} \omega(x_i - x_j)\right)^2 k(k - K_{ij}) + \sum_i \lambda_i^2\right) \varphi_{rs}^{[f]} = 0 \tag{28}$$

in the hyperbolic case ($c^2 > 0$); similar results are obtained in the rational ($c^2 = 0$) and trigonometric ($c^2 < 0$) cases. Hence, we did prove that from any solution of type (3), (26) of the KZ equations (1), with f_{ij} given in (11), we can obtain eigenfunctions $\varphi_{rs}^{[f]}$ of the CSM Hamiltonians [1–3] with exchange terms [13], which are characterized by any given permutational symmetry $[f]$ under particle coordinate exchange. To obtain wavefunctions describing an N-boson (N-fermion) system, it only remains to combine $\varphi_{rs}^{[f]}$ with a spin function transforming under the same (conjugate) irreducible representation $[f]$ ($[\tilde{f}]$) under exchange of the spin variables. A similar result is valid for the Hamiltonian with delta-function interactions [21], corresponding to the functions f_{ij} given in (13).

As a last point, we would like to mention that when restricting ourselves to solutions satisfying (26) or

$$K_{ij} \Phi = P^{(ij)} \Phi \qquad 1 \leqslant i < j \leqslant N \tag{29}$$

with K_{ij} defined in (27), equations (5) and (1) become equivalent to

$$\partial_i \Phi_\sigma = \left(\sum_{j \neq i} \left(f(x_i - x_j) + c\, \tau_\sigma^{(ij)}\right) K_{ij} + \lambda_{\sigma(i)}\right) \Phi_\sigma \qquad i = 1, 2, \ldots, N \tag{30}$$

and

$$\partial_i \Phi = \left(\sum_{j \neq i} \left(f(x_i - x_j) + c\, \hat{T}^{(ij)} \right) K_{ij} + \lambda^{(i)} \right) \Phi \qquad i = 1, 2, \ldots, N \tag{31}$$

respectively. In (31), $\hat{T}^{(ij)}$ is an operator whose action on functions (3) is given by

$$\hat{T}^{(ij)} \Phi = \sum_{\sigma} \tau_{\sigma}^{(ij)} \Phi_{\sigma} e_{\sigma}. \tag{32}$$

The corresponding operator $\hat{T}$ on $V \otimes V$ may be taken as

$$\hat{T} = \sum_{k>l} (E_{kk} \otimes E_{ll} - E_{ll} \otimes E_{kk}). \tag{33}$$

4. Conclusion

In the present paper, we have moved one step further towards a deeper understanding of the interplay between integrable systems and KZ equations (and, therefore, conformal models). We indeed showed that the results of Matsuo and Cherednik can be generalized to provide wavefunctions, characterized by any given permutational symmetry, for some CSM models with exchange terms, once solutions of the corresponding KZ equations are known. Such models include the spin generalizations of the original Calogero and Sutherland models, as well as that with δ-function interactions.

Some interesting open questions are whether similar results may also hold true for elliptic CSM models and for integrable models related to root systems of algebras different from $\mathcal{A}_{N-1}$. The use of methods similar to those employed in [22] to construct generalizations of Dunkl operators might prove to be helpful in finding proper answers.

Acknowledgment

The author would like to thank T Brzeziński for some helpful discussions.

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